2.1 Linear Diff. Eqs.; Method of Integrating Factors

1st-order linear eqs: $P(t)\frac{dy}{dt} + Q(t)y = G(t)$

cannot contain y

if $P(t) \neq 0$, we can make the coefficient 1

$$\frac{dy}{dt} + p(t)y = g(t)$$ "standard form"

no y

Some can be solved with calculus

for example, $\underbrace{t\frac{dy}{dt} + y}_{\text{notice this is}} = 1$

$$\frac{d}{dt}(ty) = t\frac{dy}{dt} + y\cdot 1$$

product rule

So, eq. becomes

$$\frac{d}{dt}(t\,y) = 1$$

integrate with respect to t

$$ty = t + C$$

Solve for y

$$\boxed{y = 1 + \frac{C}{t}} \qquad (t \neq 0)$$

unfortunately, the left side doesn't usually directly turn into the deriv. of a product

for example, $t\frac{dy}{dt} + 3y = 1$ left side is NOT deriv. of a product (try it!)

rewrite:

$$\frac{dy}{dt} + \frac{3}{t} y = \frac{1}{t}$$

multiply both sides by t^3 (why? explained soon)

$$\underbrace{t^3 \frac{dy}{dt} + 3y^2 y}_{\frac{d}{dt}(t^3 y)} = t^2$$

$$\frac{d}{dt}(t^3 y) = t^2$$

integrate

$$t^3 y = \frac{1}{3} t^3 + C$$

$$\boxed{y = \frac{1}{3} + \frac{C}{t^3}}$$

that t^3 we multiplied by is called the <u>integrating factor</u>

(different for every equation)

How to find the integrating factor?

First, make sure leading coefficient is 1.

$$\frac{dy}{dt} + p(t)y = g(t)$$

multiply by the integrating factor $\mu(t)$ such that the left side is the derivative of $\mu(t)$ ~~and~~ times y

$$\underbrace{\mu \frac{dy}{dt} + p\mu y}_{\text{turn into}} = \mu g$$

$$\frac{d}{dt}(\mu y) = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y$$

so,
$$\mu \frac{dy}{dt} + \boxed{p\mu} y = \mu \frac{dy}{dt} + \boxed{\frac{d\mu}{dt}} y$$

equal

so, $\frac{d\mu}{dt} = p\mu$ this is a diff. eq. solve for $\mu(t)$

$$\frac{1}{\mu}\frac{d\mu}{dt} = p$$

from calculus, left side is $\frac{d}{dt} \ln|\mu|$

$$\frac{d}{dt} \ln|\mu| = p$$

integrate

$$\ln|\mu| = \int p \, dt + c$$

$$|\mu| = e^{\int p(t)dt + c} = e^{\int p(t)dt} \cdot e^{c}$$

$$\mu = \underbrace{\boxed{\pm e^{c}}}_{= C} \cdot e^{\int p \, dt}$$

$$= C e^{\int p \, dt}$$ choose ANY $C \neq 0$, so choose $C = 1$

$$\boxed{\mu(t) = e^{\int p(t)\, dt}}$$

How to solve $y' + p(t)y = g(t)$

1. find integrating factor $\mu(t) = e^{\int p(t)dt}$ works only if leading coeff = 1

2. multiply both sides of $y' + p(t)y = g(t)$ by $\mu(t)$

3. turn left side into $\frac{d}{dt}(\mu y)$

4. integrate and solve

example : $ty' + 2y = \frac{\cos(t)}{t}$ $y(\pi) = 0$, $t > 0$

make leading coeff. 1

$$y' + \underbrace{\boxed{\frac{2}{t}}}_{p(t)} y = \underbrace{\boxed{\frac{\cos(t)}{t^2}}}_{g(t)}$$

find integrating factor

$$\mu = e^{\int p\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|}$$

$$= e^{\ln(t^2)} = t^2$$

Simplify μ ALL THE WAY

multiply both sides of $y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}$ by μ

$$\underbrace{t^2 y' + 2t\,y}_{\substack{\text{if } \mu \text{ is correct} \\ \text{should turn into}}} = \cos(t)$$

$$\frac{d}{dt}(t^2 y) \qquad \text{check: } t^2 y' + 2t\,y$$

μ

now diff. eq. becomes

$$\frac{d}{dt}(t^2 y) = \cos(t)$$

integrate

$$t^2 y = \sin(t) + C$$

$$y = \frac{\sin(t)}{t^2} + \frac{C}{t^2}$$

general solution

(infinitely-many solutions because of unknown C)

we ~~we~~ were given $y(\pi) = 0$

use it to find C

plug in $t = \pi$, $y = 0$ solve for C

$$0 = \frac{\sin(\pi)}{\pi^2} + \frac{C}{\pi^2}$$

$$= 0 + \frac{C}{\pi^2}$$

so, $C = 0$

$$\boxed{y(t) = \frac{\sin(t)}{t^2}}$$

particular solution

(C is known)